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A renormalisation group method for iterated maps on the real axis

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Abstract. A renormalisation group method for one-dimensional iterated maps displaying period doublings is studied. The renormalisation transformation involves variable rescaling, similar to the doubling transformation of Feigenbaum, but is defined on the space of parameters of a given mapping. Numerical calculations are carried out for maps with different analytic dependence near the extremum.

1. Introduction

The connection between the universality of metric properties of iterated one-dimensional maps exhibiting period doubling bifurcations and the renormalisation group (RG) scheme has originally been shown by Feigenbaum (1978, 1979). Besides the numerical and analytical renormalisation treatment of Feigenbaum, there are also other RG methods, both for one-dimensional (Derrida *et al* 1979, Hu and Mao 1982, Hauser *et al* 1984) and for higher-dimensional dissipative (Helleman 1980, Feigenbaum *et al* 1982) and conservative (Helleman 1980, Derrida and Pomeau 1980, Widom and Kadanoff 1982, Jansen and Tjon 1983) maps. Generally these RG procedures can be applied to those non-linear mappings which reveal chaotic behaviour and display universal scaling properties at the onset of chaos.

The renormalisation transformation (RT) presented in this paper is defined on the space of parameters of a particular function, contrary to the Feigenbaum method in which the RG flow is considered in a space of functions (see Collet and Eckmann 1980, Eckmann 1984). The idea of examining the RG flow in the space of parameters of a given map has already been used by Derrida *et al* (1979). However, the construction of RT described here differs essentially from that applied in their approach. The main difference is that our procedure involves explicitly the rescaling of the variable and does not require the existence of any pointwise homeomorphism.

2. The renormalisation group procedure

Consider the following family of mappings of the real axis:

$$g_\lambda : x \to x' = g(\{a^{(i)}(\lambda)\}, x) \tag{2.1}$$

with

$$g(\{a^{(i)}(\lambda)\}, x) = 1 - \sum_{i=1}^{l} a^{(i)}(\lambda)|x|^{zi} \tag{2.2}$$

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where $z \in \mathbb{R}$, $z > 1$, and each of the elements of the set of coefficients $a^{(i)}(\lambda)$, $i = 1, 2, \ldots, l$, is a differentiable function of a non-linearity parameter λ. The functions $g_\lambda(x)$ are symmetric and they will all be assumed to possess a unique maximum at $x = 0$. Let us also assume that the maps g_λ exhibit infinite cascades of period doubling bifurcations as the parameter λ increases to the critical value λ_∞. Here, we restrict our attention to bifurcation sequences corresponding to R^{*n} MSS sequences (Metropolis *et al* 1973, Collet and Eckmann 1980) and representing superstable periods. Since the universal metric properties of iterated maps have a local character (Feigenbaum 1978, 1979), the RG analysis of the universality in the behaviour of non-linear maps can be carried out only in a local manner. Below, we consider two versions of a RG method, the first for $x \simeq 0$ (the centred renormalisation) and the second for $x \simeq x_0$, where x_0 is a non-zero point to be determined (the non-centred renormalisation).

2.1. Centred renormalisation

Define for a given set of coefficients $\{a^{(i)}(\lambda)\}$ the functions

$$g_{n,p,r}(x) = (-\alpha)^n g^{(2^n)}\left(\{a_p^{(i)}(\lambda_{n+p+r})\}, \frac{x}{\alpha^n}\right) \tag{2.3}$$

with $n, p, r = 0, 1, 2, \ldots$, $g^{(2^n)}$ being the 2^n-fold functional composition ($g \circ g \circ \ldots \circ g$ (2^n times)), and α denoting the universal rescaling factor determined by the condition $\lim_{n,p,r\to\infty} g_{n,p,r}(0) = 1$. The coefficients $a_p^{(i)}(\lambda_{n+p+r})$, $i = 1, 2, \ldots, l$, are effective parameters (after 2^p compositions), i.e.

$$\begin{aligned} a_p^{(i)}(\lambda_r) &= -\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}|x|^i} g_{p,0,r-p}(|x|^{1/z})|_{x=0} \\ a_0^{(i)}(\lambda_r) &\equiv a^{(i)}(\lambda_r) \qquad i = 1, 2, \ldots, l \end{aligned} \tag{2.4}$$

where λ_r is the value of λ at which the 2^r-point superstable limit cycle (i.e. the set $\{\bar{x}_j\}$, $j = 1, 2, \ldots, 2^r$, such that $g_{\lambda_r}^{(2^r)}(\bar{x}_j) = \bar{x}_j$ and $(\mathrm{d}/\mathrm{d}\bar{x}_j) g_{\lambda_r}^{(2^r)}(\bar{x}_j) = 0$) occurs. Since the functional composition generates new terms in the functions $g_{n,p,r}(x)$, it follows that, in general, $g_{n+1,p,r}(x) \neq g_{n,p+1,r}(x)$.

Using equation (2.3) we construct a RT, which consists of the following four steps.

(I) Start at $\lambda = \lambda_{n+p+r+1}$ and look at $g_{n,p,r+1}(x)$ near $x = 0$.

(II) Form the functional composition:

$$g_{n,p,r+1}^{(2)}(x) = g_{n,p,r+1}(g_{n,p,r+1}(x)).$$

(III) Rescale:

$$g_{n,p,r+1}^{(2)}(x) \to -\alpha g_{n,p,r+1}^{(2)}(x/\alpha) = g_{n+1,p,r}(x).$$

(IV) Change the coefficients:

$$g_{n+1,p,r}(x) \to g_{n+1,p-1,r}(x).$$

Thus, our RT can be written in the form

$$T_\alpha \tilde{g}_{n,p,r+1}(x) = \tilde{g}_{n+1,p-1,r}(x) \tag{2.5}$$

$$\tilde{g}_{n+1,p-1,r}(x) = -\alpha \tilde{g}_{n,p,r+1}(\tilde{g}_{n,p,r+1}(x/\alpha)). \tag{2.6}$$

The essence of the transformation T_α may be explained as follows. The starting function $g_{n,p,r+1}(x)$ is associated with $\lambda = \lambda_{n+p+r+1}$ and displays a superstable cycle of

period 2^{r+1}. This function is converted by operations II and III into a new function with λ of the same value, but having a superstable cycle of period 2^r. The operation IV decreases λ from $\lambda_{n+p+r+1}$ to λ_{n+p+r} preserving the period of the superstable orbit. Thus, T_α transforms the initial function which displays a superstable 2^{r+1} cycle at $\lambda_{n+p+r+1}$ into a similar function which has a superstable 2^r cycle at λ_{n+p+r}. Consequently, equation (2.6) is recursively unstable. It is to be noted that the operation IV is not an identity transformation, provided that p remains finite. Accordingly, we denote the renormalised coefficients of the functions $\tilde{g}_{n,p,r}(x)$ by $a^{(i)}_{n,p}(\lambda_{n+p+r})$ with $n>0$ and $i=1,2,\ldots,l_n$ where $l_n>l$ ($l_n=\infty$ for all $n>0$ when z is non-integer). In principle, T_α has the form of the doubling transformation (Feigenbaum 1979). However, according to the conventional formulation, the doubling transformation does not involve step IV, i.e. it leaves the parameter λ unchanged. The reason for introducing the operation IV is to simplify our analysis of the RG flow near the fixed point.

It should be pointed out that the operator T_α acts on a function space. However, elements of limit cycles associated with the renormalised functions $\tilde{g}_{n,p,r}(x)$ are determined only by the coefficients $a^{(i)}_{n,p}(\lambda_{n+p+r})$. Therefore, the RG analysis of the universal properties of g_λ at the onset of chaos can be carried out within a space of the coefficients of $\tilde{g}_{n,p,r}(x)$. We notice that, for non-integer z and $n>0$, the functions $\tilde{g}_{n,p,r}(x)$ involve infinitely many parameters, as in such a case the functional composition of any reduced function (2.2) (with finite l) generates infinitely many higher-order terms ($l_n=\infty$). However, for integer z and all finite n, each of the renormalised functions $\tilde{g}_{n,p,r}(x)$ involves a finite number of parameters (if l is finite), although T_α generates new higher-order terms. (Note that $l_n\to\infty$ as $n\to\infty$.) Thus, in this case the dimension of the parameter space is not invariant under T_α. Obviously, one can expect that, for asymptoptically large n, p, and r, $\tilde{g}_{n,p,r}(x)$ are good approximations, even for finite l_n, to the limit function $\lim_{n,p,r\to\infty}\tilde{g}_{n,p,r}(x)$ which involves infinitely many parameters (cf Feigenbaum 1979). In any case, we can restrict concrete calculations to the parameter space of a finite dimension. This is fairly justified, since the transformation T_α is determined locally near $x=0$, and the renormalised functions $\tilde{g}_{n,p,r}(x)$ should also be considered in the immediate vicinity of $x=0$.

We assume equations (2.5) and (2.6) to have a fixed point given by

$$\begin{aligned}\lim_{n,p,r\to\infty} a^{(i)}_{n,p}(\lambda_{n+p+r+1}) &= \lim_{n,p,r\to\infty} a^{(i)}_{n+1,p-1}(\lambda_{n+p+r}) = a^{(i)*}(\lambda_\infty)\\ \lim_{n,p,r\to\infty} \tilde{g}_{n,p,r+1}(x) &= \lim_{n,p,r\to\infty} \tilde{g}_{n+1,p-1,r}(x) = g(x)\end{aligned} \tag{2.7}$$

and by the condition

$$g(0)=1. \tag{2.8}$$

Then, from (2.6) one obtains

$$g(x)=-\alpha g(g(x/\alpha)) \tag{2.9}$$

where $g(x)$ is the universal function (see Kawai and Tye 1984). According to (2.8), the scaling factor α can be determined by the formula

$$\alpha^{-1}=-g(1) \tag{2.10}$$

or equivalently by

$$\alpha^{z-1}=-\frac{\mathrm{d}}{\mathrm{d}x}g(x)|_{x=1}.$$

It is seen from the above definitions and equations (2.5) and (2.6) that the repeated application of our RT does not affect the factor α. Consequently, as long as n, p and r are finite, the value of $\tilde{g}_{n,p,r}(x)$ at $x=0$ changes under T_α. It should be noted that the relation (2.9) is exactly the equation derived by Feigenbaum and Cvitanović (Feigenbaum 1978) for the infinite attractor. Obviously, the analysis of the behaviour of the functions $\tilde{g}_{n,p,r}(x)$ near $g(x)$ is quite different from the Feigenbaum treatment. We remark that a unique analytic solution of the functional equation (2.9) has been proven to exist in the case when $z=1+\varepsilon$ with ε taking on small positive values (Collet *et al* 1980) and in the non-trivial case of $z=2$ (see Lanford 1984). In the latter case, the proof has a computational character.

Using (2.6) yields the RG equations for $a^{(i)}_{n,p}(\lambda_{n+p+r})$. These equations can formally be expressed as

$$a^{(i)}_{n+1,p-1}(\lambda_{n+p+r}) = M_i(\{a^{(k)}_{n,p}(\lambda_{n+p+r+1})\}) \qquad i=1,2,\ldots,l_n. \tag{2.11}$$

Defining for large n, p and r the deviations

$$\begin{aligned}\varepsilon^{(i)}_{n+1,p-1}(\lambda_{n+p+r}) &= a^{(i)}_{n+1,p-1}(\lambda_{n+p+r}) - a^{(i)^*}(\lambda_\infty)\\ \varepsilon^{(i)}_{n,p}(\lambda_{n+p+r+1}) &= a^{(i)}_{n,p}(\lambda_{n+p+r+1}) - a^{(i)^*}(\lambda_\infty)\end{aligned} \tag{2.12}$$

and linearising equations (2.11) about the fixed point, one obtains

$$\varepsilon^{(i)}_{n+1,p-1}(\lambda_{n+p+r}) = \sum_{j=1}^{l_n} M_{ij}(\{a^{(k)^*}(\lambda_\infty)\})\varepsilon^{(j)}_{n,p}(\lambda_{n+p+r+1}) \tag{2.13}$$

where

$$M_{ij}(\{a^{(k)^*}(\lambda_\infty)\}) = \frac{\partial M_i(\{a^{(k)^*}(\lambda_\infty)\})}{\partial a^{(j)^*}(\lambda_\infty)}. \tag{2.14}$$

The deviations $\varepsilon^{(i)}_{n+1,p-1}(\lambda_{n+p+r})$ and $\varepsilon^{(i)}_{n,p}(\lambda_{n+p+r+1})$ can be written as linear combinations of the eigenvectors $\Psi_k = (\psi^{(1)}_k, \psi^{(2)}_k, \ldots, \psi^{(l_n)}_k)$, $k=1,2,\ldots,l_n$, of the matrix M_{ij}. Then we have

$$\begin{aligned}\varepsilon^{(i)}_{n+1,p-1}(\lambda_{n+p+r}) &= \sum_{k=1}^{l_n} \gamma^{(k)}_{n+1,p-1}(\lambda_{n+p+r})\psi^{(i)}_k\\ \varepsilon^{(i)}_{n,p}(\lambda_{n+p+r+1}) &= \sum_{k=1}^{l_n} \gamma^{(k)}_{n,p}(\lambda_{n+p+r+1})\psi^{(i)}_k.\end{aligned} \tag{2.15}$$

Assuming that $\gamma^{(k)}_{n+1,p-1}(\lambda_{n+p+r})$ and $\gamma^{(k)}_{n,p}(\lambda_{n+p+r+1})$ are smooth functions of λ_{n+p+r} and $\lambda_{n+p+r+1}$, and vanish when $n, p, r=\infty$, one can expand them

$$\begin{aligned}\gamma^{(k)}_{n+1,p-1}(\lambda_{n+p+r}) &= A^{(k)}_{n+1,p-1}(\lambda_{n+p+r}-\lambda_\infty) + B^{(k)}_{n+1,p-1}(\lambda_{n+p+r}-\lambda_\infty)^2+\ldots\\ \gamma^{(k)}_{n,p}(\lambda_{n+p+r+1}) &= A^{(k)}_{n,p}\delta^{-1}_{n+p+r}(\lambda_{n+p+r}-\lambda_\infty) + B^{(k)}_{n,p}\delta^{-2}_{n+p+r}(\lambda_{n+p+r}-\lambda_\infty)^2+\ldots\end{aligned} \tag{2.16}$$

where the coefficients A and B are independent of λ (we assume that all the A are non-zero) and

$$\delta_{n+p+r} = \frac{\lambda_{n+p+r}-\lambda_\infty}{\lambda_{n+p+r+1}-\lambda_\infty}. \tag{2.17}$$

For asymptotically large n, one has

$$\delta_{n+p+r} = \delta + \Delta_{n+p+r} \tag{2.18}$$

with $\delta = \lim_{n\to\infty} \delta_{n+p+r}$ being the universal bifurcation ratio and $\Delta_{n+p+r} \to 0$ as $n \to \infty$. It should be noted that the coefficients A diverge if any of the indices n, p, r remain finite while the others tend to infinity, since the fixed-point solution to equation (2.6) can exist only if n, p and r are simultaneously infinite. Accordingly, the expansions (2.16) make sense only for large n, p, r, such that $0 < p/n < \infty$ and $0 < r/n < \infty$. Then, by equations (2.13), (2.15), (2.16) and (2.18) one derives

$$A^{(k)}_{n+1,p-1} = \rho_k \delta^{-1} A^{(k)}_{n,p} \tag{2.19}$$

where ρ_k, $k = 1, 2, \ldots, l_n$, are the eigenvalues of the matrix M_{ij}, associated with the eigenvectors $\boldsymbol{\Psi}_k$, $k = 1, 2, \ldots, l_n$. The 'group' property of the transformation (2.5) ($T_\alpha T_\alpha \boldsymbol{\Psi}_k = T_{\alpha^2} \boldsymbol{\Psi}_k$) implies that

$$A^{(k)}_{n+1,p-1} = \alpha^{y_k} A^{(k)}_{n,p} \tag{2.20}$$

with y_k being independent of α. By calling y_1 the greatest of the exponents y_k, and Λ the greatest of the eigenvalues of M_{ij}, one obtains from equations (2.15), (2.16) and (2.20)

$$\varepsilon^{(i)}_{n+1,p-1}(\lambda_{n+p+r}) = \alpha^{y_1} \delta \gamma^{(1)}_{n,p}(\lambda_{n+p+r+1}) \psi^{(i)}_1 + O(\alpha^{y_2}, \Delta_{n+p+r}) \tag{2.21}$$

where y_2 is greater than all the other y_k. Thus, we obtain

$$\Lambda = \alpha^{y_1} \delta. \tag{2.22}$$

Since the RG equations (2.11) are recursively unstable, we have $\Lambda > 1$. It should be pointed out that the matrix M_{ij} is not symmetric and its eigenvalues do not have to be real.

It follows from (2.22) that, to calculate the universal constant δ, one has to determine the exponent y_1. This can be done as follows. Using (2.6) and (2.13) we find that

$$\varepsilon^{(i)}_{n+1,p-1}(\lambda_{n+p+r}) = \alpha^{1-zi} \sum_{j=1}^{l_n} L_{ij}(\delta) \varepsilon^{(j)}_{n,p}(\lambda_{n+p+r+1}) \tag{2.23}$$

where

$$L_{ij}(\delta) = \alpha^{zi-1} M_{ij}(\{a^{(k)*}(\lambda_\infty)\}) \qquad i, j = 1, 2, \ldots, l_n \tag{2.24}$$

is independent of α. Accordingly, the bifurcation ratio δ can be expressed as

$$\delta = \max_{1 \le k \le l_n} \{\eta_k\} \tag{2.25}$$

with η_k being real roots of the equation

$$\det(L_{ij}(\delta) - \eta \delta_{i,j}) = 0. \tag{2.26}$$

Consequently, one obtains

$$y_1 = 1 - z \qquad y_2 = 1 - 2z \tag{2.27}$$

etc. Thus, all the y_k are real. By virtue of (2.22), Λ must be real and positive.

We note that the greatest eigenvalue of linearised RG equations appearing both in the method of Feigenbaum (1979) and the method of Derrida *et al* (1979) does not depend on α and is equal to δ. As concerns the former approach, this follows from the fact that a resulting linearised RG functional equation is satisfied, after an appropriate separation, by an eigenvector being exactly the generator of infinitesimal rescaling. The latter approach does not involve explicitly the variable rescaling at all. Thus, our RG method yields a different procedure of calculating the universal constants.

2.2. Non-centred renormalisation

The RT can also be constructed for $x \simeq x_0$ with $x_0 > 0$ (Feigenbaum 1978, Derrida *et al* 1979). In this case one considers the following non-linear functions

$$g(\{a^{(i)}(\lambda)\}, x) = 1 - \sum_{i=1}^{l} a^{(i)}(\lambda) x^{z+i-1}. \tag{2.28}$$

The functions $g_{n,p,r}(x)$ are now defined by

$$g_{n,p,r}(x) = (-1)^{n-1} \alpha^{z+n} [g^{(2^n)}(\{a_p^{(i)}(\lambda_{n+p+r})\}, x_{n,p} + \sigma_z x/\alpha^{z+n-1}) - x_{n,p}] \tag{2.29}$$

with

$$\begin{aligned} a_p^{(i)}(\lambda_r) &= -\frac{1}{i!} \frac{\mathrm{d}^i}{\mathrm{d}x^i} g_{p,0,r-p}(x^{1/z})\bigg|_{x=0} \\ a_0^{(i)}(\lambda_r) &\equiv a^{(i)}(\lambda_r) \qquad i = 1, 2, \ldots, l \end{aligned} \tag{2.30}$$

and $x_{n,p} > 0$ being the position of the maximum of $g^{(2)}_{n,p,r}(x)$. This position is determined by

$$g_{n,p,r}(x_{n,p}) = 0. \tag{2.31}$$

It is to be noted that, because of the condition (2.31), $x_{n,p}$ depends on the parameter λ. The factor σ_z, given by

$$\sigma_z = \begin{cases} 1 & \text{if } z = 2, 4, \ldots \\ -1 & \text{otherwise} \end{cases}$$

ensures that the functions $g_{n,p,r}(x)$ are real for any real value of z.

The first two steps of the RT for $x \simeq x_0$ are analogous to those introduced for the case of centred renormalisation, except that the functions $g_{n,p,r+1}(x)$ are considered now near x_0. The remaining steps of the non-centred RT are as follows.

(III) Shift the system of coordinates:

$$g^{(2)}_{n,p,r+1}(x) \to g^{(2)}_{n,p,r+1}(x_{n,p} + \sigma_z x) - x_{n,p}.$$

(IV) Rescale:

$$g^{(2)}_{n,p,r+1}(x_{n,p} + \sigma_z x) - x_{n,p} \to \alpha^z [g^{(2)}_{n,p,r+1}(x_{n,p} + \sigma_z x/\alpha^z) - x_{n,p}] = g_{n+1,p,r+1}(x).$$

(V) Change the coefficients:

$$g_{n+1,p,r+1}(x) \to g_{n+1,p-1,r}(x).$$

Thus, in the present case, the RT takes the form

$$T_\alpha \tilde{g}_{n,p,r+1}(x) = \tilde{g}_{n+1,p-1,r}(x) \tag{2.32}$$

$$\tilde{g}_{n+1,p-1,r}(x) = \alpha^z [\tilde{g}_{n,p,r+1}(\tilde{g}_{n,p,r+1}(x_{n,p} + \sigma_z x/\alpha^z)) - x_{n,p}]. \tag{2.33}$$

The fixed-point solution to equation (2.33) fulfils the relation

$$g(x) = \alpha^z [g(g(x_0 + \sigma_z x/\alpha^z)) - x_0] \qquad x \simeq x_0 \tag{2.34}$$

and the conditions

$$g(0) = 1 \qquad g(x_0) = 0$$

where $x_0 = x_{\infty,\infty}$. The scaling factor α is determined by

$$\alpha^{-z} = 1 - x_0 \tag{2.35}$$

or equivalently by

$$\alpha^{z-1} = -\frac{\mathrm{d}}{\mathrm{d}x} g(x)\bigg|_{x=x_0}. \tag{2.36}$$

According to (2.33), the linearised RG equations have the form

$$\varepsilon^{(i)}_{n+1,p-1}(\lambda_{n+p+r}) = \sum_{j=1}^{l_n} M_{ij}(\{a^{(k)*}(\lambda_\infty)\}, \{\partial_k x_0\}, x_0)\varepsilon^{(j)}_{n,p}(\lambda_{n+p+r+1}) \tag{2.37}$$

where

$$\partial_k x_0 = \lim_{n,p,r\to\infty} \frac{\partial x_{n,p}(\lambda_{n+p+r+1})}{\partial a^{(k)}_{n,p}(\lambda_{n+p+r+1})}. \tag{2.38}$$

Thus, in order to find the eigenvalues of the matrix M_{ij}, one has to calculate the derivatives $\partial_k x_0$. This can be performed by taking into account the change of $\tilde{g}_{n,p,r+1}(0)$ under the RT. Using (2.31) and (2.33), one can express this change by the relation

$$\alpha^z(\tilde{g}_{n,p,r+1}(0) - \tilde{g}_{n,p,r+1}(x_{n,p}) - x_{n,p}) = \tilde{g}_{n+1,p-1,r}(0) - \tilde{g}_{n+1,p-1,r}(x_{n+1,p-1}). \tag{2.39}$$

Then the derivatives $\partial_k x_0$ can be found by differentiating the above equation with respect to $a^{(k)}_{n,p}(\lambda_{n+p+r+1})$, using (2.37), and taking the limit $n, p, r \to \infty$. In general, such a procedure yields many solutions for each of the derivatives $\partial_k x_0$. Obviously, the appropriate solutions are those for which eigenvalues of M_{ij} take on the greatest values.

It is to be noted that, in the case of non-centred renormalisation, the functional composition does not change the value of the function $g(x)$ at $x = 0$, i.e. $g(0) = g(g(x_0))$. The rescaling which appears in the construction of the non-centred RT (step IV) is due only to shifting of the system of coordinates (step III). Consequently, the greatest of the exponents y_k is given now by $y_1 = 0$. Thus, in the case of the non-centred RG procedure, the greatest eigenvalue of the matrix M_{ij} is

$$\Lambda = \delta. \tag{2.40}$$

It must be stressed that the RG approach described in this paper does not involve any pointwise homeomorphism, which would enable one to preserve the dimension of the parameter space. This is in contrast with the method of Derrida *et al* (1979), based on the following conjugacy law $g_A = h^{-1} \circ g_a \circ g_a \circ h$, where the mapping g is, in general, of the form (2.1) and (2.2), and A, a denote parameter sets of the same number of elements. As emphasised by Derrida *et al*, the homeomorphism h cannot, in general, be a polynomial or a rational function, and its existence cannot even be proved. In practice, this homeomorphism may only be determined approximately, in the form of polynomials, and then the conjugacy law is not strictly satisfied. Thus, the RG scheme presented here appears to be more coherent than that of Derrida *et al.*

One should notice that, in the method of Feigenbaum (1978, 1979), the RG flow near the fixed point is analysed in a space of infinitely many times composed functions. Consequently, in this approach the renormalised functions have, in principle, the form of infinite series, but are defined at $\lambda = \lambda_\infty$. Recently, the Feigenbaum RG scheme has been extended to the non-linearity parameter λ away from its accumulation value (Liu *et al* 1984).

2.3. The surface of criticality

It is to be noted that, in general, the RT of the form (2.6) (or (2.33)) may have many fixed points. In particular, there should always exist non-trivial fixed points for which

$a^{(i)*}(\lambda_\infty)=0$, $i=1,2,\ldots,l'$ with $l'<l_n$. Such fixed points correspond to universal functions $g(x)$ with $z'>z$, where z' and z are powers determining the behaviour of the maxima of $g(x)$ and $g_\lambda(x)$, respectively. For a given fixed point of this type, the matrix M_{ij} can have many eigenvalues $|\rho_k|>1$. This is connected with the possibility of occurring relevant fields (see Niemeijer and van Leeuwen 1976) for which $a^{(i)}_{n,p}(\lambda_{n+p+r})\neq 0$ $(i\leqslant l')$ when $p<\infty$ and/or $r<\infty$. Clearly, the eigenvalues $|\rho_k|<1$ are related to the RG flow within the critical surface. It is obvious that the critical surface is associated with $\lambda_\infty=\lambda_\infty(z')$. According to equation (2.6) (or (2.33)), this corresponds to $p=\infty$ and $r=\infty$. Note that for finite p and/or r equation (2.6) (or (2.33)) describes the RG flow in an unstable direction. Taking into account that $\gamma^{(k)}_{n,p}(\lambda_{n+p+r})$ depend on λ_{n+p+r} only through the coefficients $a^{(i)}_{n,p}(\lambda_{n+p+r})$, the surface of criticality can generally be determined by

$$\gamma^{(k)}_{n,\infty}(\lambda_\infty(z'))=0 \qquad k\leqslant l'$$

where k indexes those $\gamma^{(k)}_{n,\infty}$ which correspond to the successive eigenvalues $|\rho_k|>1$.

3. Numerical results

We restrict ourselves to numerical calculations of the universal constants δ and α for several chosen values of z. The calculations are performed for approximate renormalisations, keeping lower-order terms with several parameters in the functions $g_{n,p,r}(x)$. As an example, we present some details of our procedure for one-parameter approximations. In the case of the centred renormalisation and $z=2$, we obtain

$$a^{(1)}_{n+1,p-1}(\lambda_{n+p+r})=2\alpha^{-1}[a^{(1)}_{n,p}(\lambda_{n+p+r+1})]^2$$
$$\alpha=[a^{(1)*}(\lambda_\infty)-1]^{-1}$$

and

$$\Lambda=2.$$

The non-centred renormalisation gives, for $z=2$,

$$a^{(1)}_{n+1,p-1}(\lambda_{n+p+r})=4\alpha^{-2}x^2_{n,p}[a^{(1)}_{n,p}(\lambda_{n+p+r+1})]^3$$
$$a^{(1)*}(\lambda_\infty)x_0^2=1 \tag{3.1}$$

and

$$\alpha=(1-x_0)^{-1/2}.$$

The relation (2.39) yields

$$\alpha^2(a^{(1)}_{n,p}(\lambda_{n+p+r+1})x^2_{n,p}-x_{n,p})=a^{(1)}_{n+1,p-1}(\lambda_{n+p+r})x^2_{n+1,p-1}. \tag{3.2}$$

Next, differentiating equation (3.2) and using (3.1), one derives

$$(\alpha/x_0)^2(\partial_1 x_0)^2+[4\alpha-\alpha^2(\alpha-1)]\partial_1 x_0-(\alpha^2-3)x_0^2=0.$$

Finally, on linearising equation (3.1), the greatest eigenvalue is found to be

$$\Lambda=\delta=4.414\,214.$$

It should be noted that, within analogous approximations, our method leads to the same fixed-point solutions of the RG equations, and thereby to the same values for α (in the case of centred renormalisation) as the procedure described by Derrida *et al*

(1979). Obviously, this does not concern the so-called 'equality-of-slopes' RG procedure also developed by these authors. As regards the calculation of δ, our method gives generally better results than those given by Derrida *et al.* The detailed results which we have obtained for $z = 2$ and $z = 4$ are shown in table 1. To examine more reliably the efficiency of our method, we have applied the one-parameter approximation to cases of other values of z (see table 2). The values given in tables 1 and 2 appear to suggest that for large z the non-centred renormalisation is more accurate than the centred one. It should be noted, however, that the calculations carried out by means

Table 1. Values of the universal constants obtained by using the RG method for the set of R^{*n} sequences. The errors are calculated in relation to exact (numerical) values (see Feigenbaum 1978, Hu and Satija 1983).

z	RG procedure l_n	δ	Error in δ (%)	α	Error in α (%)
2	Centred renormalisation				
	1	5.464 102	17.0	2.732 051	9.2
	2	4.892 837	4.8	2.534 030	1.2
	3	4.631 825	0.8	2.478 909	1.0
	Non-centred renormalisation				
	1	4.414 214	5.5	2.414 214	3.5
	2	4.721 514	1.1	2.514 239	0.5
4	Centred renormalisation				
	1	12.359 467	69.5	1.835 087	8.6
	2	9.826 797	34.8	1.732 354	2.5
	3	9.291 572	27.5	1.737 977	2.8
	Non-centred renormalisation				
	1	6.989 756	4.0	1.663 252	1.6
	2	7.552 850	3.7	1.701 412	0.6

Table 2. Universal constants obtained within the one-parameter approximation. The relative errors are calculated using exact values given by Derrida *et al* (1979) and Hu and Satija (1983).

z	RG procedure	δ	Error in δ (%)	α	Error in α (%)
	Centred renormalisation				
1.1		2.468 336	12.9	8.198 668	2.4
1.5		3.821 640	0.6	3.651 233	7.7
3		5.675 860	6.7	1.684 616	12.6
6		19.610 989	111	1.578 534	7.5
8		19.835 336	81	1.387 855	2.8
	Non-centred renormalisation				
1.1		2.780 540	1.9	7.805 402	2.6
1.5		4.100 921	7.9	3.263 240	3.7
3		6.312 569	3.7	1.879 385	2.5
6		9.310 987	0.01	1.462 197	0.4
8		13.650 018	24	1.362 879	0.9

of the non-centred procedure are much more involved. It is also remarkable that for growing z the accuracy of calculations decreases and the convergence of results obtained in successive approximations becomes slower. Finally, we remark that our method can easily be extended to m-furcation sequences with $m = 3, 4, \ldots$.

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